Prophet Inequalities via Balanced Prices

Thomas Kesselheim

Institut für Informatik
Universität Bonn

EC'21 Tutorial

Online Combinatorial Auction

- n buyers, arriving one by one
- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

- n buyers, a

$$v_1(\{1\}) = 1$$
$$v_1(\{2\}) = 2$$
$$v_1(\{1,2\}) = 2$$

- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

$$v_2(\{1\}) = 0$$
$$v_2(\{2\}) = 10$$
$$v_2(\{1, 2\}) = 10$$

- n buyers, arriving one b

0

- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

$$v_2(\{1\}) = 0$$
$$v_2(\{2\}) = 10$$
$$v_2(\{1,2\}) = 10$$

- n buyers, arriving one b

0

10

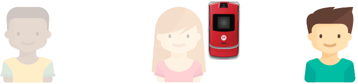
- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

$$v_3(\{1\}) = 5$$
$$v_3(\{2\}) = 5$$
$$v_3(\{1,2\}) = 5$$

- n buyers, arriving one by one



0 10

- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

- n buyers, arriving one by one

0 10 5

$$v_3(\{1\}) = 5$$
$$v_3(\{2\}) = 5$$
$$v_3(\{1,2\}) = 5$$

- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

- n buyers, arriving one by one

$$v_4(\{1\}) = 20$$
$$v_4(\{2\}) = 50$$
$$v_4(\{1,2\}) = 80$$

0 10 5

- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

- *n* buyers, arriving one by one

0 10 5 0

- *m* items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Outline

1. Single Item
2. Balanced Prices for Combinatorial Auctions
3. Other Feasibility Structures

Outline

1. Single Item
2. Balanced Prices for Combinatorial Auctions
3. Other Feasibility Structures

Single Item: Classic Prophet Inequality

- n buyers, arriving one by one
- one item

- Goal simplifies to: Stop sequence at highest number

Single Item: Classic Prophet Inequality

- n buyers, arriving one by one

$v_1 = 15$

- one item

- Goal simplifies to: Stop sequence at highest number

Single Item: Classic Prophet Inequality

- n buyers, arriving one by one

$v_1 = 15$

$v_2 = 10$

- one item

- Goal simplifies to: Stop sequence at highest number

Single Item: Classic Prophet Inequality

- n buyers, arriving one by one

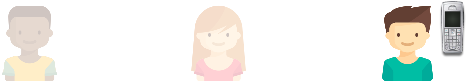
$v_1 = 15$ $v_2 = 10$ $v_3 = 30$

- one item

- Goal simplifies to: Stop sequence at highest number

Single Item: Classic Prophet Inequality

- n buyers, arriving one by one



$v_1 = 15$ $v_2 = 10$ $v_3 = 30$

- one item

- Goal simplifies to: Stop sequence at highest number

Single Item: Classic Prophet Inequality

- n buyers, arriving one by one

$v_1 = 15$ $v_2 = 10$ $v_3 = 30$ $v_4 = 80$

- one item

- Goal simplifies to: Stop sequence at highest number

Single Item: Classic Prophet Inequality

- n buyers, arriving one by one

$v_1 = 15$ $v_2 = 10$ $v_3 = 30$ $v_4 = 80$

- one item

- Goal simplifies to: Stop sequence at highest number

Compare performance to $v(\text{OPT}) = \max_i v_i$

The Prophet Inequality

- n buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- one item
- buyers arrive online, one after the other, reveal v_i, immediately decide whether to assign item or keep it

- $v_i \sim \mathcal{D}_i$ independently
- buyers arrive in order $1, 2, \ldots, n$

The Prophet Inequality

- n buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- one item
- buyers arrive online, one after the other, reveal v_i, immediately decide whether to assign item or keep it

- $v_i \sim \mathcal{D}_i$ independently
- buyers arrive in order $1, 2, \ldots, n$

Theorem (Krengel, Sucheston, Garling, 1978)

There is algorithm with

$$\mathbf{E}[v(\mathrm{ALG})] \geq \frac{1}{2}\mathbf{E}[v(\mathrm{OPT})]$$

Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]

$v_1 \sim \mathcal{D}_1$

$v_2 \sim \mathcal{D}_2$

$v_3 \sim \mathcal{D}_3$

$v_4 \sim \mathcal{D}_4$

$v_5 \sim \mathcal{D}_5$

Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]

$v_1 \sim \mathcal{D}_1$ $v_2 \sim \mathcal{D}_2$ $v_3 \sim \mathcal{D}_3$ $v_4 \sim \mathcal{D}_4$ $v_5 \sim \mathcal{D}_5$

Define any price p

Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]

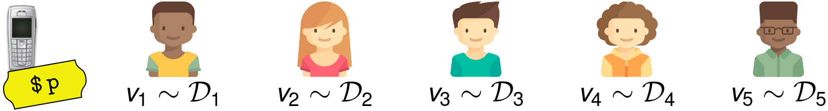
$v_1 \sim \mathcal{D}_1$ $v_2 \sim \mathcal{D}_2$ $v_3 \sim \mathcal{D}_3$ $v_4 \sim \mathcal{D}_4$ $v_5 \sim \mathcal{D}_5$

Define any price p

q: Probability that item is sold

Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]



Define any price p

q: Probability that item is sold

How much money do we collect?

$$\mathbf{E}[\text{revenue}] = pq$$

Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]

$v_1 \sim \mathcal{D}_1$ $v_2 \sim \mathcal{D}_2$ $v_3 \sim \mathcal{D}_3$ $v_4 \sim \mathcal{D}_4$ $v_5 \sim \mathcal{D}_5$

Define any price p

q: Probability that item is sold

How much money do we collect?

$$\mathbf{E}[\text{revenue}] = pq$$

What is a buyer's utility (value minus payment)?

$$\begin{aligned}
\mathbf{E}[u_i] &= \mathbf{E}[(v_i - p)^+ \cdot \mathbf{1}_{\text{nobody before } i \text{ buys}}] \\
&= \mathbf{E}[(v_i - p)^+] \cdot \mathbf{Pr}[\text{nobody before } i \text{ buys}] \\
&\geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)
\end{aligned}$$

Putting the Pieces Together

So far:

$$\mathbf{E}[\text{revenue}] = pq \qquad \text{and} \qquad \mathbf{E}[u_i] \geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)$$

Putting the Pieces Together

So far:

$$\mathbf{E}[\text{revenue}] = pq \qquad \text{and} \qquad \mathbf{E}[u_i] \geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)$$

In combination:

$$\mathbf{E}[\text{welfare}] = \mathbf{E}[\text{revenue}] + \sum_i \mathbf{E}[u_i]$$

Putting the Pieces Together

So far:

$$\mathbf{E}[\text{revenue}] = pq \qquad \text{and} \qquad \mathbf{E}[u_i] \geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)$$

In combination:

$$\begin{aligned} \mathbf{E}[\text{welfare}] &= \mathbf{E}[\text{revenue}] + \sum_i \mathbf{E}[u_i] \\ &\geq pq + \sum_i \mathbf{E}[(v_i - p)^+] \cdot (1 - q) \end{aligned}$$

Putting the Pieces Together

So far:

$$\mathbf{E}[\text{revenue}] = pq \qquad \text{and} \qquad \mathbf{E}[u_i] \geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)$$

In combination:

$$\begin{aligned}
\mathbf{E}[\text{welfare}] &= \mathbf{E}[\text{revenue}] + \sum_i \mathbf{E}[u_i] \\
&\geq pq + \sum_i \mathbf{E}[(v_i - p)^+] \cdot (1 - q) \\
&\geq pq + \mathbf{E}[\max_i v_i - p] \cdot (1 - q)
\end{aligned}$$

Putting the Pieces Together

So far:

$$\mathbf{E}[\text{revenue}] = pq \qquad \text{and} \qquad \mathbf{E}[u_i] \geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)$$

In combination:

$$\begin{aligned} \mathbf{E}[\text{welfare}] &= \mathbf{E}[\text{revenue}] + \sum_i \mathbf{E}[u_i] \\ &\geq pq + \sum_i \mathbf{E}[(v_i - p)^+] \cdot (1 - q) \\ &\geq pq + \mathbf{E}[\max_i v_i - p] \cdot (1 - q) \end{aligned}$$

For $p = \frac{1}{2}\mathbf{E}[\max_i v_i]$, this yields

$$\mathbf{E}[\text{welfare}] \geq \frac{1}{2}\mathbf{E}[\max_i v_i]q + \frac{1}{2}\mathbf{E}[\max_i v_i](1 - q) = \frac{1}{2}\mathbf{E}[\max_i v_i]$$

The Essence

$v_1 = 10$

$v_2 = 30$

$v_3 = 15$

$v_4 = 80$

$v_5 = 5$

Consider full information.

$v_1 = 10$

$v_2 = 30$

$v_3 = 15$

$v_4 = 80$

$v_5 = 5$

Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”

The Essence

$v_1 = 10$ $v_2 = 30$ $v_3 = 15$ $v_4 = 80$ $v_5 = 5$

Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”

Let $v_i = \max_k v_k$

The Essence

$v_1 = 10$ $v_2 = 30$ $v_3 = 15$ $v_4 = 80$ $v_5 = 5$

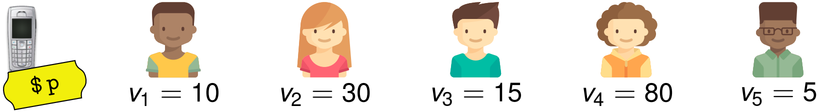
Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is "balanced"

Let $v_i = \max_k v_k$

- **Case 1:** Somebody $i' < i$ buys item
  $\Rightarrow$ revenue $\geq \frac{1}{2} v_i$

The Essence



Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”

Let $v_i = \max_k v_k$

- **Case 1:** Somebody $i' < i$ buys item
  $\Rightarrow$ revenue $\geq \frac{1}{2} v_i$
- **Case 2:** Nobody $i' < i$ buys item
  $\Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i$

$v_1 = 10$ $v_2 = 30$ $v_3 = 15$ $v_4 = 80$ $v_5 = 5$

Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”

Let $v_i = \max_k v_k$

- **Case 1:** Somebody $i' < i$ buys item
  $\Rightarrow$ revenue $\geq \frac{1}{2} v_i$
- **Case 2:** Nobody $i' < i$ buys item
  $\Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i$

In either case: welfare $=$ revenue $+$ utilities $\geq \frac{1}{2} v_i$

Outline

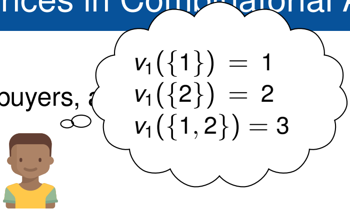
Posted Prices in Combinatorial Auctions

- n buyers, arriving one by one, $v_i \sim \mathcal{D}_i$ independently; $\mathcal{D}_i$ known in advance
- m items

- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

Posted Prices in Combinatorial Auctions

- n buyers, a $\sim \mathcal{D}_i$ independently; $\mathcal{D}_i$ known in advance



- m items

- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

Posted Prices in Combinatorial Auctions

$$v_2(\{1\}) = 0$$
$$v_2(\{2\}) = 10$$
$$v_2(\{1,2\}) = 10$$

- n buyers, arriving one b… …ndently; $\mathcal{D}_i$ known in advance

0

- m items

- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

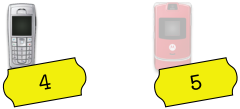
Posted Prices in Combinatorial Auctions

$$v_2(\{1\}) = 0$$
$$v_2(\{2\}) = 10$$
$$v_2(\{1,2\}) = 10$$

- n buyers, arriving one b… …ndently; $\mathcal{D}_i$ known in advance

0 10

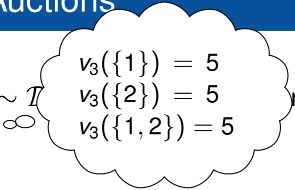
- m items



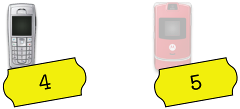
- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

Posted Prices in Combinatorial Auctions

- n buyers, arriving one by one, $v_i \sim \mathcal{D}$ ... nown in advance



- m items



- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

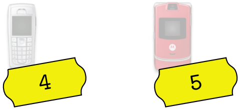
Posted Prices in Combinatorial Auctions

- n buyers, arriving one by one, $v_i \sim \mathcal{I}$ … nown in advance

$$v_3(\{1\}) = 5$$
$$v_3(\{2\}) = 5$$
$$v_3(\{1,2\}) = 5$$

0 10 5

- m items

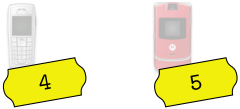


- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

Posted Prices in Combinatorial Auctions

- n buyers, arriving one by one, $v_i \sim \mathcal{D}_i$ independently; $\mathcal{D}_i$ known in advance

- m items



- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

XOS Combinatorial Auctions

A function v_i is XOS or fractionally subadditive if there are $v_{i,j}^{\ell} \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^{\ell} \quad .$$

XOS Combinatorial Auctions

A function v_i is XOS or fractionally subadditive if there are $v_{i,j}^{\ell} \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^{\ell} \quad .$$

Examples:

- additive $v_i(S) = \sum_{j \in S} v_i(\{j\})$

XOS Combinatorial Auctions

A function v_i is XOS or fractionally subadditive if there are $v_{i,j}^{\ell} \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^{\ell} \quad .$$

Examples:

- additive $v_i(S) = \sum_{j \in S} v_i(\{j\})$
- unit-demand $v_i(S) = \max_{j \in S} v_i(\{j\})$

XOS Combinatorial Auctions

A function v_i is XOS or fractionally subadditive if there are $v_{i,j}^{\ell} \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^{\ell} \quad .$$

Examples:

- additive $v_i(S) = \sum_{j \in S} v_i(\{j\})$
- unit-demand $v_i(S) = \max_{j \in S} v_i(\{j\})$
- budget-additive $v_i(S) = \min \left\{ B, \sum_{j \in S} v_{i,j} \right\}$

XOS Combinatorial Auctions

A function v_i is XOS or fractionally subadditive if there are $v_{i,j}^{\ell} \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^{\ell} \quad .$$

Examples:

- additive $v_i(S) = \sum_{j \in S} v_i(\{j\})$
- unit-demand $v_i(S) = \max_{j \in S} v_i(\{j\})$
- budget-additive $v_i(S) = \min \left\{ B, \sum_{j \in S} v_{i,j} \right\}$
- submodular $v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T)$ for $S \subseteq T$

Prophet Inequality for XOS Combinatorial Auctions

[Feldman, Gravin, Lucier SODA 2015]

Theorem (Feldman, Gravin, Lucier SODA 2015)

For any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ over XOS functions, there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot \mathbf{E}[OPT(v)].$$

Prophet Inequality for XOS Combinatorial Auctions

[Feldman, Gravin, Lucier SODA 2015]

Theorem (Feldman, Gravin, Lucier SODA 2015)

For any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ over XOS functions, there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot \mathbf{E}[OPT(v)].$$

Generalizes classic prophet inequality.

Prophet Inequality for XOS Combinatorial Auctions

[Feldman, Gravin, Lucier SODA 2015]

Theorem (Feldman, Gravin, Lucier SODA 2015)

For any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ over XOS functions, there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot \mathbf{E}[OPT(v)].$$

Generalizes classic prophet inequality.

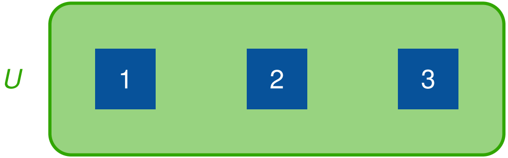
Key Technique: Balanced Prices

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U)$$

$$U = \{1, 2, 3\}$$



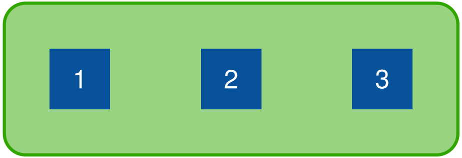
Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \;\; (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \;\; (\forall S \subseteq U)$$

$U = \{1, 2, 3\}$

U



Example 1: Additive

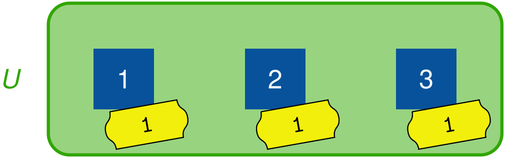
$v_1(S) = |S|$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \ \ (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \ \ (\forall S \subseteq U)$$

$U = \{1, 2, 3\}$



Example 1: Additive

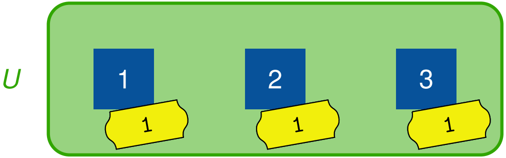
$v_1(S) = |S|$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \ (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \ (\forall S \subseteq U)$$ ✓

$U = \{1, 2, 3\}$



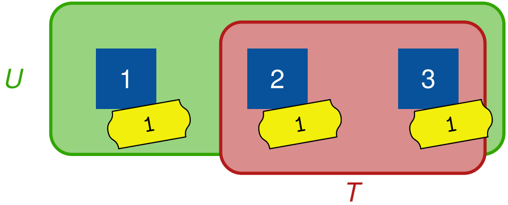
Example 1: Additive

$v_1(S) = |S|$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \;\; (\forall T \subseteq U) \quad \checkmark \qquad \sum_{j \in S} p_j \leq v_i(S) \;\; (\forall S \subseteq U) \quad \checkmark$$

$$U = \{1, 2, 3\}$$



Example 1: Additive

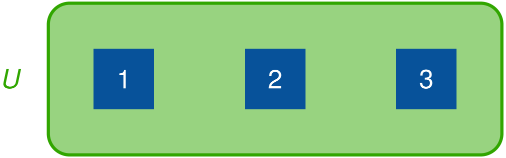
$$v_1(S) = |S|$$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \;\; (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \;\; (\forall S \subseteq U)$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

Example 2: Unit-Demand

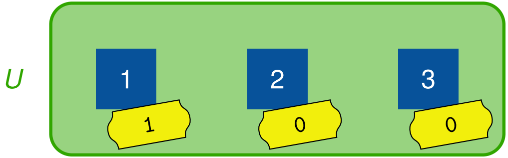
$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U)$$

$$U = \{1, 2, 3\}$$



Example 1: Additive

$$v_1(S) = |S|$$

Example 2: Unit-Demand

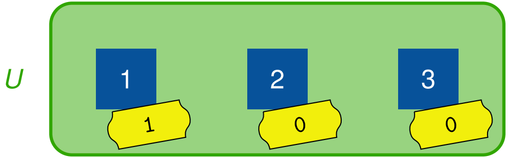
$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \;\; (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \;\; (\forall S \subseteq U) \quad \checkmark$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

Example 2: Unit-Demand

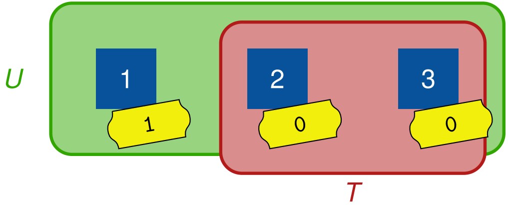
$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \ \ (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \ \ (\forall S \subseteq U) \quad \checkmark$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

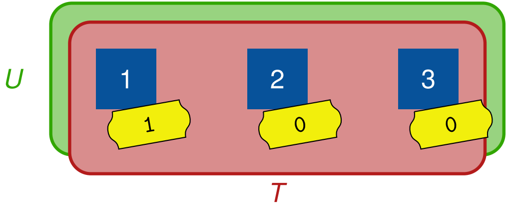
Example 2: Unit-Demand

$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \qquad \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \quad \checkmark$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

Example 2: Unit-Demand

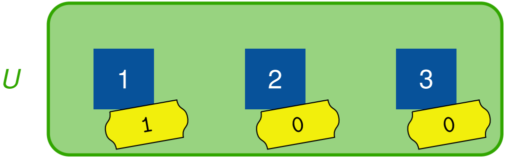
$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \quad \checkmark$$

$$\sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \quad \checkmark$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

Example 2: Unit-Demand

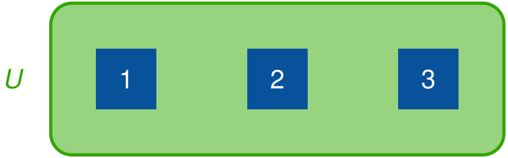
$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \;\; (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \;\; (\forall S \subseteq U)$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

Example 2: Unit-Demand

$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Example 3: Budget-Additive

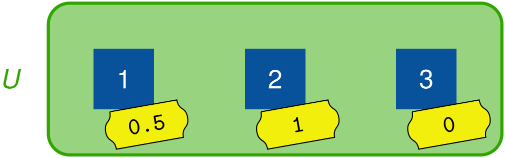
$v_3(S) = \min\{|S|, 1.5\}$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U)$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

Example 2: Unit-Demand

$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Example 3: Budget-Additive

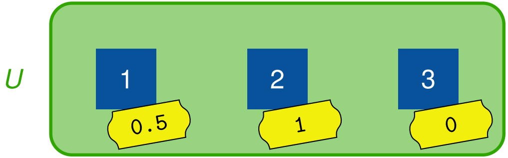
$v_3(S) = \min\{|S|, 1.5\}$

Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \;\; (\forall T \subseteq U) \quad \checkmark$$

$$\sum_{j \in S} p_j \leq v_i(S) \;\; (\forall S \subseteq U) \quad \checkmark$$

$U = \{1, 2, 3\}$



Example 1: Additive

$v_1(S) = |S|$

Example 2: Unit-Demand

$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Example 3: Budget-Additive

$v_3(S) = \min\{|S|, 1.5\}$

Balanced Prices: General Approximation Bound

[Dütting, Feldman, K., Lucier, FOCS 2017] and earlier work

Definition

A valuation function v_i admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices p_j for $j \in U$ such that for all $T \subseteq U$:

(a) $\sum_{j \in T} p_j \geq (v_i(U) - v_i(U \setminus T))$ (b) $\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T)$

Balanced Prices: General Approximation Bound

[Dütting, Feldman, K., Lucier, FOCS 2017] and earlier work

Definition

A valuation function v_i admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices p_j for $j \in U$ such that for all $T \subseteq U$:

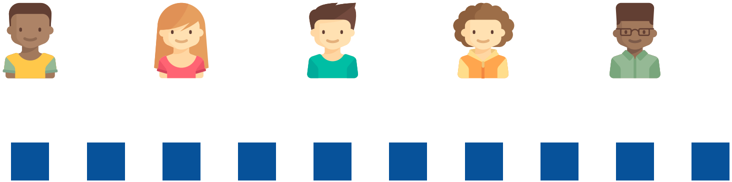
(a) $\sum_{j \in T} p_j \geq \quad (v_i(U) - v_i(U \setminus T))$ (b) $\sum_{j \in U \setminus T} p_j \leq \quad v_i(U \setminus T)$

Theorem

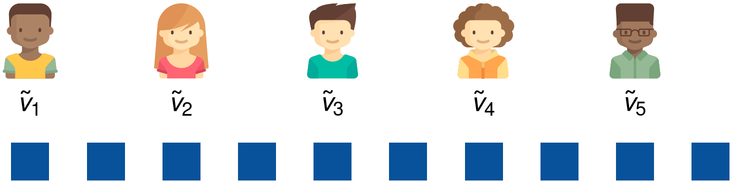
If a class of valuations admits balanced prices, then for any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot \mathbf{E}[OPT].$$

Setting the Prices

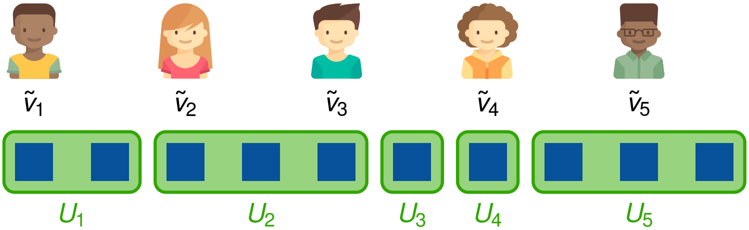


Setting the Prices



Fix $\tilde{v}_1, \ldots, \tilde{v}_n$

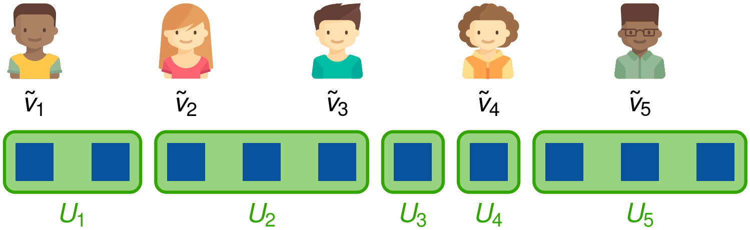
Setting the Prices



Fix $\tilde{v}_1, \ldots, \tilde{v}_n$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(\tilde{v})\}$

Setting the Prices

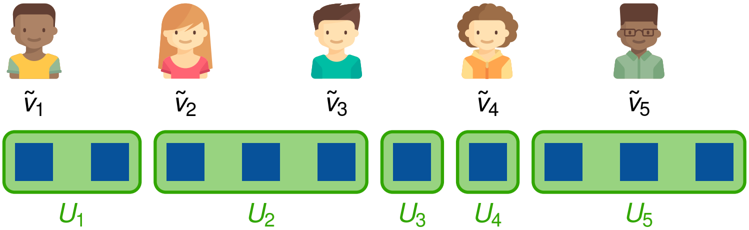


Fix $\tilde{v}_1, \ldots, \tilde{v}_n$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(\tilde{v})\}$

For $j \in U_i$ set $p_j^{\tilde{v}}$ to balanced price for item j in $\tilde{v}_i$, U_i

Setting the Prices



Fix $\tilde{v}_1, \ldots, \tilde{v}_n$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(\tilde{v})\}$

For $j \in U_i$ set $p_j^{\tilde{v}}$ to balanced price for item j in $\tilde{v}_i$, U_i

Price for item j: $\bar{p}_j = \frac{1}{2}\mathbf{E}_{\tilde{v}\sim\mathcal{D}}[p_j^{\tilde{v}}]$.

Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

$$\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)$$

$$\sum_{j \in U_i \setminus T_i} p_j \leq v_i(U_i \setminus T_i)$$

Proof Sketch Full Information

$$\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)$$
$$\sum_{j \in U_i \setminus T_i} p_j \leq v_i(U_i \setminus T_i)$$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left(v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j$$

Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

$$\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)$$

$$\sum_{j \in U_i \setminus T_i} p_j \leq v_i(U_i \setminus T_i)$$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left(v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j$$

$$\geq \left(v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left(v_i(U_i) - v_i(U_i \setminus T_i) \right)$$

Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)$$

$$\sum_{j \in U_i \setminus T_i} p_j \leq v_i(U_i \setminus T_i)$$

$$u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left(v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j$$

$$\geq \left(v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left(v_i(U_i) - v_i(U_i \setminus T_i) \right)$$

$$= \frac{1}{2} v_i(U_i)$$

Proof Sketch Full Information

$$\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)$$

$$\sum_{j \in U_i \setminus T_i} p_j \leq v_i(U_i \setminus T_i)$$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$\sum_{i=1}^{n} v_i(X_i) \geq \sum_{i=1}^{n} \left[u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \right] \geq \sum_{i=1}^{n} \left[\left(v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j \right]$$

$$\geq \sum_{i=1}^{n} \left[\left(v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left(v_i(U_i) - v_i(U_i \setminus T_i) \right) \right]$$

$$= \sum_{i=1}^{n} \frac{1}{2} v_i(U_i)$$

Balanced Prices: General Approximation Bound

[Dütting, Feldman, K., Lucier, FOCS 2017] and earlier work

Definition

A valuation function v_i admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices p_j for $j \in U$ such that for all $T \subseteq U$:

(a) $\sum_{j \in T} p_j \geq (v_i(U) - v_i(U \setminus T))$ (b) $\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T)$

Theorem

If a class of valuations admits balanced prices, then for any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot \mathbf{E}[OPT].$$

Balanced Prices: General Approximation Bound

[Dütting, Feldman, K., Lucier, FOCS 2017] and earlier work

Definition

A valuation function v_i admits (α, β)-balanced prices if for every set of items $U \subseteq [m]$ there exist item prices p_j for $j \in U$ such that for all $T \subseteq U$:

(a) $\sum_{j \in T} p_j \geq \frac{1}{\alpha} (v_i(U) - v_i(U \setminus T))$ (b) $\sum_{j \in U \setminus T} p_j \leq \beta v_i(U \setminus T)$

Theorem

If a class of valuations admits balanced prices, then for any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{1 + \alpha\beta} \cdot \mathbf{E}[OPT].$$

Combinatorial Auctions: Further Results

- Prices can be computed in polynomial time via LP relaxation (instead of integral optimum)

Combinatorial Auctions: Further Results

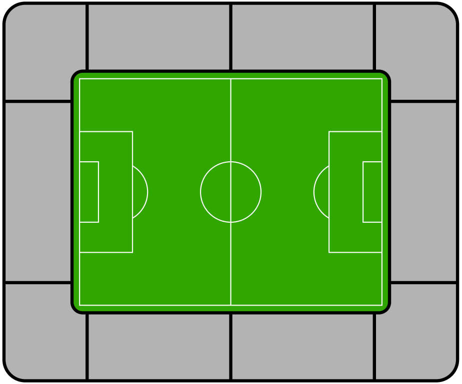
- Prices can be computed in polynomial time via LP relaxation (instead of integral optimum)
- Techniques also applicable for revenue maximization [Cai and Zhao, STOC 2017]

Combinatorial Auctions: Further Results

- Prices can be computed in polynomial time via LP relaxation (instead of integral optimum)
- Techniques also applicable for revenue maximization [Cai and Zhao, STOC 2017]
- Subadditive functions v_i only admit (α, β)-balanced prices for $\alpha\beta = \Omega(\log m)$ but better bounds via relaxation of balancedness [Dütting, K., Lucier, FOCS 2020]

Outline

Matroid Prophet Inequality: Example

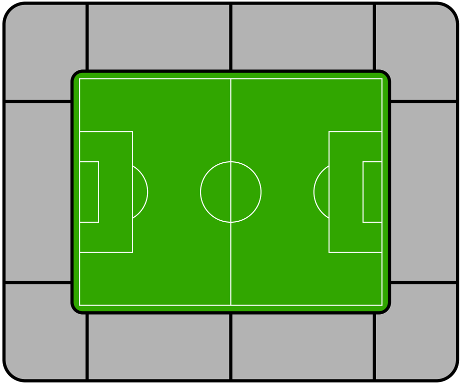


May assign. . .

$\leq$ 100 seats in each of 10 blocks

$\leq$ 500 seats in entire stadium

Matroid Prophet Inequality: Example



May assign...

$\leq$ 100 seats in each of 10 blocks

$\leq$ 500 seats in entire stadium

Laminar matroid

Matroid Prophet Inequality: Problem Statement

[Kleinberg and Weinberg, STOC 2012], somewhat implicit in earlier work

- n buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- set of accepted buyers has to be independent in given matroid
- buyers arrive online, one after the other, reveal v_i, immediately decide whether to accept or to reject current buyer

- $v_i \sim \mathcal{D}_i$ independently
- buyers arrive in order $1, 2, \ldots, n$

Matroid Prophet Inequality: Problem Statement

[Kleinberg and Weinberg, STOC 2012], somewhat implicit in earlier work

- n buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- set of accepted buyers has to be independent in given matroid
- buyers arrive online, one after the other, reveal v_i,
  immediately decide whether to accept or to reject current buyer

- $v_i \sim \mathcal{D}_i$ independently
- buyers arrive in order $1, 2, \ldots, n$

Theorem (Kleinberg, Weinberg, STOC 2012)

There is algorithm with

$$\mathbf{E}\left[v(\mathrm{ALG})\right] \geq \frac{1}{2}\mathbf{E}\left[v(\mathrm{OPT})\right]$$

Generalized Balanced Prices

[Dütting, Feldman, K., Lucier, FOCS 2017], similar approach in [Kleinberg and Weinberg, STOC 2012]

Definition

A pricing rule $p^{\mathbf{v}}$ is (α, β)-balanced with respect to valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ if for all feasible $\mathbf{x}$ and all $\mathbf{x}'$ that are feasible "after" $\mathbf{x}$

(a) $\sum_i p_i^{\mathbf{v}}(x_i \qquad) \geq \frac{1}{\alpha}\left(\mathbf{v}\left(\mathrm{OPT}(\mathbf{v})\right) - \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))\right)$

(b) $\sum_i p_i^{\mathbf{v}}(x_i' \qquad) \leq \beta\, \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))$

$\mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))$: Value that remains after allocating to $\mathbf{x}$

$\mathbf{v}\left(\mathrm{OPT}(\mathbf{v})\right) - \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))$: Value lost due to allocating to $\mathbf{x}$

Generalized Balanced Prices

[Dütting, Feldman, K., Lucier, FOCS 2017], similar approach in [Kleinberg and Weinberg, STOC 2012]

Definition

A pricing rule $p^{\mathbf{v}}$ is (α, β)-balanced with respect to valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ if for all feasible $\mathbf{x}$ and all $\mathbf{x}'$ that are feasible "after" $\mathbf{x}$

(a) $\sum_i p_i^{\mathbf{v}}(x_i \mid \mathbf{x}_{[i-1]}) \geq \frac{1}{\alpha} \left(\mathbf{v}\left(\mathrm{OPT}(\mathbf{v})\right) - \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))\right)$

(b) $\sum_i p_i^{\mathbf{v}}(x_i' \mid \mathbf{x}_{[i-1]}) \leq \beta\, \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))$

$\mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))$: Value that remains after allocating to $\mathbf{x}$

$\mathbf{v}\left(\mathrm{OPT}(\mathbf{v})\right) - \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))$: Value lost due to allocating to $\mathbf{x}$

Generalized Balanced Prices

[Dütting, Feldman, K., Lucier, FOCS 2017], similar approach in [Kleinberg and Weinberg, STOC 2012]

Definition

A pricing rule $p^{\mathbf{v}}$ is (α, β)-balanced with respect to valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ if for all feasible $\mathbf{x}$ and all $\mathbf{x}'$ that are feasible "after" $\mathbf{x}$

(a) $\sum_i p_i^{\mathbf{v}}(x_i \mid \mathbf{x}_{[i-1]}) \geq \frac{1}{\alpha} \left(\mathbf{v}\left(\mathrm{OPT}(\mathbf{v})\right) - \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))\right)$

(b) $\sum_i p_i^{\mathbf{v}}(x_i' \mid \mathbf{x}_{[i-1]}) \leq \beta \, \mathbf{v}(\mathrm{OPT}\left(\mathbf{v} \mid \mathbf{x}\right))$

Theorem

If $p^{\mathbf{v}}$ is (α, β)-balanced for every $\mathbf{v}$, then setting $p_i(x_i \mid \mathbf{y}) = \frac{\alpha}{1+\alpha\beta} \mathbf{E}_{\tilde{\mathbf{v}} \sim D}\left[p_i^{\tilde{\mathbf{v}}}(x_i \mid \mathbf{y})\right]$ achieves welfare at least $\frac{1}{1+\alpha\beta} \mathbf{E}[\mathbf{v}\left(\mathrm{OPT}(\mathbf{v})\right)]$.

Example: Matroids

[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: $p_i^{\tilde{v}}(\text{accept} \mid A) = \text{OPT}(\tilde{v}, \mathcal{M}/A) - \text{OPT}(\tilde{v}, \mathcal{M}/(A \cup \{i\})$, where

- A is the set of buyers accepted so far
- $\mathcal{M}/A$ is the matroid obtained by contracting A

Example: Matroids

[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: $p_i^{\tilde{v}}(\text{accept} \mid A) = \text{OPT}(\tilde{v}, \mathcal{M}/A) - \text{OPT}(\tilde{v}, \mathcal{M}/(A \cup \{i\}))$, where

- A is the set of buyers accepted so far
- $\mathcal{M}/A$ is the matroid obtained by contracting A

One can show: $(1, 1)$-balanced

Example: Matroids

[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: $p_i^{\tilde{v}}(\text{accept} \mid A) = \text{OPT}(\tilde{v}, \mathcal{M}/A) - \text{OPT}(\tilde{v}, \mathcal{M}/(A \cup \{i\})$, where

- A is the set of buyers accepted so far
- $\mathcal{M}/A$ is the matroid obtained by contracting A

One can show: $(1, 1)$-balanced

Price for buyer i: $p_i(\text{accept} \mid A) = \frac{1}{2}\mathbf{E}_{\tilde{v}}\left[\text{OPT}(\tilde{v}, \mathcal{M}/A) - \text{OPT}(\tilde{v}, \mathcal{M}/(A \cup \{i\})\right]$.

Example: Matroids

[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: $p_i^{\tilde{v}}(\text{accept} \mid A) = \text{OPT}(\tilde{v}, \mathcal{M}/A) - \text{OPT}(\tilde{v}, \mathcal{M}/(A \cup \{i\}))$, where

- A is the set of buyers accepted so far
- $\mathcal{M}/A$ is the matroid obtained by contracting A

One can show: $(1, 1)$-balanced

Price for buyer i: $p_i(\text{accept} \mid A) = \frac{1}{2}\mathbf{E}_{\tilde{v}}\left[\text{OPT}(\tilde{v}, \mathcal{M}/A) - \text{OPT}(\tilde{v}, \mathcal{M}/(A \cup \{i\})\right]$.

Prices have to depend on selections so far. [Feldman/Svensson/Zenklusen, SICOMP 2021]

Conclusion

- General advantage:
  Showing that balancedness involves only full-information arguments

- Can also be applied in prophet-secretary setting
  [Ehsani, Hajiaghayi, K., Singla, SODA 2018]

- Sometime balanced prices give optimal guarantees for prophet inequalities
  (but not always)

Conclusion

- General advantage:
  Showing that balancedness involves only full-information arguments

- Can also be applied in prophet-secretary setting
  [Ehsani, Hajiaghayi, K., Singla, SODA 2018]

- Sometime balanced prices give optimal guarantees for prophet inequalities
  (but not always)

Thank you!