Prophet Inequalities via Balanced Prices

Thomas Kesselheim

Institut für Informatik
Universität Bonn

EC’21 Tutorial
Online Combinatorial Auction

- \( n \) buyers, arriving one by one

- \( m \) items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
Online Combinatorial Auction

- \( n \) buyers,
  \[ v_1(\{1\}) = 1 \]
  \[ v_1(\{2\}) = 2 \]
  \[ v_1(\{1, 2\}) = 2 \]

- \( m \) items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
Online Combinatorial Auction

- $n$ buyers, arriving one by one
- $m$ items
- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

$v_2(\{1\}) = 0$
$v_2(\{2\}) = 10$
$v_2(\{1, 2\}) = 10$
Online Combinatorial Auction

- $n$ buyers, arriving one by one
- $m$ items
- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

$v_2(\{1\}) = 0$
$v_2(\{2\}) = 10$
$v_2(\{1, 2\}) = 10$
Online Combinatorial Auction

- $n$ buyers, arriving one by one
- $m$ items
- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

\[
\begin{align*}
\nu_1(\{1\}) &= 1 \\
\nu_1(\{2\}) &= 2 \\
\nu_1(\{1, 2\}) &= 2 \\
\nu_2(\{1\}) &= 10 \\
\nu_2(\{2\}) &= 0 \\
\nu_2(\{1, 2\}) &= 10 \\
\nu_3(\{1\}) &= 5 \\
\nu_3(\{2\}) &= 5 \\
\nu_3(\{1, 2\}) &= 5 \\
\end{align*}
\]
Online Combinatorial Auction

- $n$ buyers, arriving one by one

- $m$ items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

\[ v_1(\{1\}) = 1 \]
\[ v_1(\{2\}) = 2 \]
\[ v_1(\{1, 2\}) = 2 \]
\[ v_2(\{1\}) = 0 \]
\[ v_2(\{2\}) = 10 \]
\[ v_2(\{1, 2\}) = 10 \]
\[ v_3(\{1\}) = 5 \]
\[ v_3(\{2\}) = 5 \]
\[ v_3(\{1, 2\}) = 5 \]
Online Combinatorial Auction

- $n$ buyers, arriving one by one
- $m$ items
- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

$v_4(\{1\}) = 20$
$v_4(\{2\}) = 50$
$v_4(\{1, 2\}) = 80$
Online Combinatorial Auction

- $n$ buyers, arriving one by one

- $m$ items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
1. Single Item

2. Balanced Prices for Combinatorial Auctions

3. Other Feasibility Structures
1 Single Item

2 Balanced Prices for Combinatorial Auctions

3 Other Feasibility Structures
Single Item: Classic Prophet Inequality

- $n$ buyers, arriving one by one

- one item

- Goal simplifies to: Stop sequence at highest number
Single Item: Classic Prophet Inequality

- $n$ buyers, arriving one by one
  
  $v_1 = 15$

- one item

- Goal simplifies to: Stop sequence at highest number
Single Item: Classic Prophet Inequality

- $n$ buyers, arriving one by one

\[ v_1 = 15 \quad v_2 = 10 \]

- one item

- Goal simplifies to: Stop sequence at highest number
Single Item: Classic Prophet Inequality

- $n$ buyers, arriving one by one

\[ \begin{align*}
v_1 &= 15 \\ v_2 &= 10 \\ v_3 &= 30
\end{align*} \]

- one item

- Goal simplifies to: Stop sequence at highest number
Single Item: Classic Prophet Inequality

- $n$ buyers, arriving one by one

\[
\begin{align*}
\nu_1 &= 15 \\
\nu_2 &= 10 \\
\nu_3 &= 30
\end{align*}
\]

- one item

- Goal simplifies to: Stop sequence at highest number
Single Item: Classic Prophet Inequality

- \( n \) buyers, arriving one by one

\[
\begin{align*}
 v_1 &= 15 \\
 v_2 &= 10 \\
 v_3 &= 30 \\
 v_4 &= 80
\end{align*}
\]

- one item

- Goal simplifies to: Stop sequence at highest number
Single Item: Classic Prophet Inequality

- $n$ buyers, arriving one by one

- $v_1 = 15$
- $v_2 = 10$
- $v_3 = 30$
- $v_4 = 80$

- one item

- Goal simplifies to: Stop sequence at highest number

Compare performance to $v(OPT) = \max_i v_i$
The Prophet Inequality

- $n$ buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- one item
- buyers arrive online, one after the other, reveal $v_i$, immediately decide whether to assign item or keep it

- $v_i \sim D_i$ independently
- buyers arrive in order $1, 2, \ldots, n$
The Prophet Inequality

- $n$ buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- one item
- buyers arrive online, one after the other, reveal $v_i$, immediately decide whether to assign item or keep it

- $v_i \sim \mathcal{D}_i$ independently
- buyers arrive in order $1, 2, \ldots, n$

Theorem (Krengel, Sucheston, Garling, 1978)

There is algorithm with

$$E[\nu(\text{ALG})] \geq \frac{1}{2}E[\nu(\text{OPT})]$$
Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]

\[ v_1 \sim \mathcal{D}_1 \quad v_2 \sim \mathcal{D}_2 \quad v_3 \sim \mathcal{D}_3 \quad v_4 \sim \mathcal{D}_4 \quad v_5 \sim \mathcal{D}_5 \]
Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]

Define any price $p$

$v_1 \sim D_1 \quad v_2 \sim D_2 \quad v_3 \sim D_3 \quad v_4 \sim D_4 \quad v_5 \sim D_5$

Probability that item is sold $q$

How much money do we collect?

$E[\text{revenue}] = pq$

What is a buyer's utility (value minus payment)?

$E[u_i] = E[(v_i - p) + \cdot 1_{\text{nobody before } i \text{ buys}}]$
Define any price $p$

$q$: Probability that item is sold
Prophet Inequality

Define any price $p$
$q$: Probability that item is sold

How much money do we collect?

$$E[\text{revenue}] = pq$$
Prophet Inequality

cf. [Samuel-Cahn, 1984], [Kleinberg/Weinberg, STOC 2012]

Define any price $p$
$q$: Probability that item is sold

How much money do we collect?

$$E[\text{revenue}] = pq$$

What is a buyer's utility (value minus payment)?

$$E[u_i] = E[(v_i - p)^+ \cdot 1_{\text{nobody before } i \text{ buys}}]$$

$$= E[(v_i - p)^+] \cdot \Pr[\text{nobody before } i \text{ buys}]$$

$$\geq E[(v_i - p)^+] \cdot (1 - q)$$
So far:

\[ E[\text{revenue}] = pq \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]
Putting the Pieces Together

So far:

\[ E[\text{revenue}] = pq \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]

In combination:

\[ E[\text{welfare}] = E[\text{revenue}] + \sum_i E[u_i] \]
Putting the Pieces Together

So far:

\[ E[\text{revenue}] = pq \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]

In combination:

\[ E[\text{welfare}] = E[\text{revenue}] + \sum_i E[u_i] \]

\[ \geq pq + \sum_i E[(v_i - p)^+] \cdot (1 - q) \]
Putting the Pieces Together

So far:

\[ E[\text{revenue}] = pq \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]

In combination:

\[ E[\text{welfare}] = E[\text{revenue}] + \sum_i E[u_i] \]
\[ \geq pq + \sum_i E[(v_i - p)^+] \cdot (1 - q) \]
\[ \geq pq + E[\max_i v_i - p] \cdot (1 - q) \]
Putting the Pieces Together

So far:
\[ E[\text{revenue}] = pq \quad \text{and} \quad E[u_i] \geq E[(v_i - p)^+] \cdot (1 - q) \]

In combination:
\[
E[\text{welfare}] = E[\text{revenue}] + \sum_i E[u_i] \\
\geq pq + \sum_i E[(v_i - p)^+] \cdot (1 - q) \\
\geq pq + E[\max_i v_i - p] \cdot (1 - q)
\]

For \( p = \frac{1}{2} E[\max_i v_i] \), this yields
\[
E[\text{welfare}] \geq \frac{1}{2} E[\max_i v_i]q + \frac{1}{2} E[\max_i v_i](1 - q) = \frac{1}{2} E[\max_i v_i]
\]
Consider full information.
The Essence

Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”
Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”

Let $v_i = \max_k v_k$
Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”

Let $v_i = \max_k v_k$

- **Case 1**: Somebody $i' < i$ buys item
  \[\Rightarrow \text{revenue} \geq \frac{1}{2} v_i\]
The Essence

Consider full information.

Price \( p = \frac{1}{2} \max_k v_k \) is “balanced”

Let \( v_i = \max_k v_k \)

- **Case 1:** Somebody \( i' < i \) buys item
  \[ \Rightarrow \text{revenue} \geq \frac{1}{2} v_i \]

- **Case 2:** Nobody \( i' < i \) buys item
  \[ \Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i \]
Consider full information.

Price $p = \frac{1}{2} \max_k v_k$ is “balanced”

Let $v_i = \max_k v_k$

- **Case 1:** Somebody $i' < i$ buys item
  $\Rightarrow$ revenue $\geq \frac{1}{2} v_i$

- **Case 2:** Nobody $i' < i$ buys item
  $\Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i$

**In either case:** welfare $= \text{revenue} + \text{utilities} \geq \frac{1}{2} v_i$
Outline

1. Single Item

2. Balanced Prices for Combinatorial Auctions

3. Other Feasibility Structures
Posted Prices in Combinatorial Auctions

- $n$ buyers, arriving one by one, $v_i \sim D_i$ independently; $D_i$ known in advance

- $m$ items

- Precompute item prices $p_1, \ldots, p_m$

- At each arrival: Arriving buyer purchases utility maximizing bundle

- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$
Posted Prices in Combinatorial Auctions

- \(n\) buyers, arriving one by one, \(v_i \sim D_i\) independently; \(D_i\) known in advance

\[
\begin{align*}
\nu_1(\{1\}) &= 1 \\
\nu_1(\{2\}) &= 2 \\
\nu_1(\{1, 2\}) &= 3
\end{align*}
\]

- \(m\) items

- Precompute item prices \(p_1, \ldots, p_m\)
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare \(\sum_{i=1}^n \nu_i(X_i)\)
Posted Prices in Combinatorial Auctions

- $n$ buyers, arriving one by one, independently; $D_i$ known in advance

\[
\begin{align*}
v_2(\{1\}) &= 0 \\
v_2(\{2\}) &= 10 \\
v_2(\{1, 2\}) &= 10
\end{align*}
\]

- $m$ items

- Precompute item prices $p_1, \ldots, p_m$

- At each arrival: Arriving buyer purchases utility maximizing bundle

- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$
Posted Prices in Combinatorial Auctions

- $n$ buyers, arriving one by one independently; $D_i$ known in advance
  
  \[
  \nu_2(\{1\}) = 0 \\
  \nu_2(\{2\}) = 10 \\
  \nu_2(\{1, 2\}) = 10
  \]

- $m$ items

  - Precompute item prices $p_1, \ldots, p_m$
  - At each arrival: Arriving buyer purchases utility maximizing bundle
  - Maximize social welfare $\sum_{i=1}^n \nu_i(X_i)$

Thomas Kesselheim
Posted Prices in Combinatorial Auctions

- $n$ buyers, arriving one by one, $v_i \sim \mathcal{D}$ independently; $\mathcal{D}$ known in advance

\begin{align*}
&v_3(\{1\}) = 5 \\
&v_3(\{2\}) = 5 \\
&v_3(\{1, 2\}) = 5
\end{align*}

- $m$ items

- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$
Posted Prices in Combinatorial Auctions

- $n$ buyers, arriving one by one, $v_i \sim \mathcal{D}$ known in advance
- $m$ items
- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases utility maximizing bundle
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

\[ v_3(\{1\}) = 5 \]
\[ v_3(\{2\}) = 5 \]
\[ v_3(\{1, 2\}) = 5 \]
Posted Prices in Combinatorial Auctions

- \( n \) buyers, arriving one by one, \( v_i \sim \mathcal{D}_i \) independently; \( \mathcal{D}_i \) known in advance

- \( m \) items

- Precompute item prices \( p_1, \ldots, p_m \)

- At each arrival: Arriving buyer purchases utility maximizing bundle

- Maximize social welfare \( \sum_{i=1}^{n} v_i(X_i) \)
A function $v_i$ is XOS or fractionally subadditive if there are $v_{i,j}^\ell \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^\ell.$$
A function $v_i$ is XOS or fractionally subadditive if there are $v_{i,j}^\ell \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^\ell .$$

Examples:
- additive $v_i(S) = \sum_{j \in S} v_i(\{j\})$
A function \( v_i \) is **XOS** or fractionally subadditive if there are \( v_{i,j}^\ell \in \mathbb{R}_{\geq 0} \) such that

\[
v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^\ell.
\]

Examples:

- **additive** \( v_i(S) = \sum_{j \in S} v_i(\{j\}) \)
- **unit-demand** \( v_i(S) = \max_{j \in S} v_i(\{j\}) \)
A function \( v_i \) is \textbf{XOS} or fractionally subadditive if there are \( v_{i,j}^\ell \in \mathbb{R}_{\geq 0} \) such that

\[
v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^\ell .
\]

Examples:

- **additive** \( v_i(S) = \sum_{j \in S} v_i(\{j\}) \)
- **unit-demand** \( v_i(S) = \max_{j \in S} v_i(\{j\}) \)
- **budget-additive** \( v_i(S) = \min \{ B, \sum_{j \in S} v_{i,j} \} \)
A function $v_i$ is **XOS** or fractionally subadditive if there are $v_{i,j}^\ell \in \mathbb{R}_{\geq 0}$ such that

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^\ell .$$

Examples:

- **additive** $v_i(S) = \sum_{j \in S} v_i(\{j\})$
- **unit-demand** $v_i(S) = \max_{j \in S} v_i(\{j\})$
- **budget-additive** $v_i(S) = \min \left\{ B, \sum_{j \in S} v_{i,j} \right\}$
- **submodular** $v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T)$ for $S \subseteq T$
Theorem (Feldman, Gravin, Lucier SODA 2015)

For any distributions $D_1, \ldots, D_n$ over XOS functions, there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$, 

$$\mathbb{E} \left[ \sum_{i=1}^{n} v_i(X_i) \right] \geq \frac{1}{2} \cdot \mathbb{E}[OPT(v)].$$

Generalizes classic prophet inequality.

Key Technique: Balanced Prices
Theorem (Feldman, Gravin, Lucier SODA 2015)

For any distributions $D_1, \ldots, D_n$ over XOS functions, there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$
E \left[ \sum_{i=1}^{n} v_i(X_i) \right] \geq \frac{1}{2} \cdot E[OPT(v)].
$$

Generalizes classic prophet inequality.
Theorem (Feldman, Gravin, Lucier SODA 2015)

For any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ over XOS functions, there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$E\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot E[OPT(v)].$$

Generalizes classic prophet inequality.

Key Technique: Balanced Prices
Balanced Prices: Examples

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ U = \{1, 2, 3\} \]

\[ U \]

Example 1: Additive

\[ v_1(S) = |S| \]

Example 2: Unit-Demand

\[ v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases} \]

Example 3: Budget-Additive

\[ v_3(S) = \min\{|S|, 1.5\} \]
Balanced Prices: Examples

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive
\[ v_1(S) = |S| \]
Balanced Prices: Examples

\[
\sum_{j \in S} p_j \leq v_i(S) \ (\forall S \subseteq U)
\]

\[
\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \ (\forall T \subseteq U)
\]

\[
U = \{1, 2, 3\}
\]

Example 1: Additive

\[v_1(S) = |S|\]
Balanced Prices: Examples

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ U = \{1, 2, 3\} \]

**Example 1: Additive**

\[ v_1(S) = |S| \]

**Example 2: Unit-Demand**

\[ v_2(S) = \begin{cases} 0 & S = \emptyset \\ 1 & S \neq \emptyset \end{cases} \]

**Example 3: Budget-Additive**

\[ v_3(S) = \min\{ |S|, 1.5 \} \]
Balanced Prices: Examples

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ U = \{1, 2, 3\} \]

**Example 1: Additive**

\[ v_1(S) = |S| \]

\[ v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases} \]

\[ v_3(S) = \min\{ |S|, 1.5 \} \]
Balanced Prices: Examples

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ \sum_{j \in T} p_j \leq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive

\[ v_1(S) = |S| \]

Example 2: Unit-Demand

\[ v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases} \]
Balanced Prices: Examples

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]
\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive
\[ v_1(S) = |S| \]

Example 2: Unit-Demand
\[ v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases} \]
Balanced Prices: Examples

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive

\[ v_1(S) = |S| \]

Example 2: Unit-Demand

\[ v_2(S) = \begin{cases} 
0 & \text{if } S = \emptyset \\
1 & \text{if } S \neq \emptyset 
\end{cases} \]
Balanced Prices: Examples

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

Example 1: Additive
\[ v_1(S) = |S| \]

Example 2: Unit-Demand
\[ v_2(S) = \begin{cases} 
0 & \text{if } S = \emptyset \\
1 & \text{if } S \neq \emptyset 
\end{cases} \]

\[ U = \{1, 2, 3\} \]
Balanced Prices: Examples

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive
\[ v_1(S) = |S| \]

Example 2: Unit-Demand
\[ v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases} \]
Balanced Prices: Examples

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive
\[ v_1(S) = |S| \]

Example 2: Unit-Demand
\[ v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases} \]
Balanced Prices: Examples

$$\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U)$$

$$\sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U)$$

$$U = \{1, 2, 3\}$$

Example 1: Additive
$$v_1(S) = |S|$$

Example 2: Unit-Demand
$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

Example 3: Budget-Additive
$$v_3(S) = \min\{|S|, 1.5\}$$
Balanced Prices: Examples

\[
\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U)
\]

\[
\sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U)
\]

\[U = \{1, 2, 3\}\]

Example 1: Additive
\[v_1(S) = |S|\]

Example 2: Unit-Demand
\[v_2(S) = \begin{cases} 
0 & \text{if } S = \emptyset \\
1 & \text{if } S \neq \emptyset
\end{cases}\]

Example 3: Budget-Additive
\[v_3(S) = \min\{|S|, 1.5\}\]
Balanced Prices: Examples

\[ \sum_{j \in S} p_j \leq v_i(S) \quad (\forall S \subseteq U) \]

\[ \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \quad (\forall T \subseteq U) \]

\[ U = \{1, 2, 3\} \]

Example 1: Additive
\[ v_1(S) = |S| \]

Example 2: Unit-Demand
\[ v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases} \]

Example 3: Budget-Additive
\[ v_3(S) = \min\{|S|, 1.5\} \]
Balanced Prices: General Approximation Bound

[Dütting, Feldman, K., Lucier, FOCS 2017] and earlier work

**Definition**

A valuation function $v_i$ admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices $p_j$ for $j \in U$ such that for all $T \subseteq U$:

(a) $\sum_{j \in T} p_j \geq (v_i(U) - v_i(U \setminus T))$

(b) $\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T)$
Balanced Prices: General Approximation Bound
[Dütting, Feldman, K., Lucier, FOCS 2017] and earlier work

**Definition**
A valuation function $v_i$ admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices $p_j$ for $j \in U$ such that for all $T \subseteq U$:

(a) $\sum_{j \in T} p_j \geq (v_i(U) - v_i(U \setminus T))$
(b) $\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T)$

**Theorem**
If a class of valuations admits balanced prices, then for any distributions $D_1, \ldots, D_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$E\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot E[OPT].$$
Setting the Prices

Let $U_i = \{ j \mid i \text{ gets } j \text{ in } \text{OPT}(\tilde{v}) \}$

For $j \in U_i$ set $p_{\tilde{v}}j$ to balanced price for item $j$ in $\tilde{v}_i$.

Price for item $j$: $\bar{p}_j = \frac{1}{2} E_{\tilde{v} \sim D}[p_{\tilde{v}}j]$. 
Setting the Prices

Fix $\tilde{v}_1, \ldots, \tilde{v}_n$
Setting the Prices

Fix \( \tilde{v}_1, \ldots, \tilde{v}_n \)

Let \( U_i = \{ j \mid i \text{ gets } j \text{ in } OPT(\tilde{v}) \} \)
Fix $\tilde{v}_1, \ldots, \tilde{v}_n$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } \text{OPT}(\tilde{v})\}$

For $j \in U_i$ set $p_{\tilde{v}}^j$ to balanced price for item $j$ in $\tilde{v}_i$, $U_i$
Setting the Prices

Fix \( \tilde{v}_1, \ldots, \tilde{v}_n \)

Let \( U_i = \{ j \mid i \text{ gets } j \text{ in } OPT(\tilde{v}) \} \)

For \( j \in U_i \) set \( p_{j}^{\tilde{v}} \) to balanced price for item \( j \) in \( \tilde{v}_i, U_i \)

Price for item \( j \): \( \bar{p}_j = \frac{1}{2} E_{\tilde{v} \sim D}[p_{j}^{\tilde{v}}] \).
Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

$$\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)$$

$$\sum_{j \in U_i \setminus T_i} p_j \leq v_i(U_i \setminus T_i)$$
Proof Sketch Full Information

Let $U_i = \{ j \mid i \text{ gets } j \text{ in } \text{OPT}(\nu) \}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{ j \in U_i \text{ sold to buyers } i' \neq i \}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left( v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j$$
Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left( v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j$$

$$\geq \left( v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left( v_i(U_i) - v_i(U_i \setminus T_i) \right)$$
Proof Sketch Full Information

Let $U_i = \{ j \mid i \text{ gets } j \text{ in } OPT(v) \}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{ j \in U_i \text{ sold to buyers } i' \neq i \}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

\[
\sum_{j \in T_i} p_j \geq v_i(U_i) - v_i(U_i \setminus T_i)
\]

\[
\sum_{j \in U_i \setminus T_i} p_j \leq v_i(U_i \setminus T_i)
\]

\[
u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left( v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j \]

\[
\geq \left( v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left( v_i(U_i) - v_i(U_i \setminus T_i) \right)
\]

\[
= \frac{1}{2} v_i(U_i)
\]
Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } \text{OPT}(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U_i$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

Then, for the allocation $X_1, \ldots, X_n$, we have:

$$\sum_{i=1}^{n} v_i(X_i) \geq \sum_{i=1}^{n} \left[ u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \right] \geq \sum_{i=1}^{n} \left[ \left( v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j \right]$$

$$\geq \sum_{i=1}^{n} \left[ \left( v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left( v_i(U_i) - v_i(U_i \setminus T_i) \right) \right]$$

$$= \sum_{i=1}^{n} \frac{1}{2} v_i(U_i)$$
Balanced Prices: General Approximation Bound
[Dütting, Feldman, K., Lucier, FOCS 2017] and earlier work

Definition

A valuation function $v_i$ admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices $p_j$ for $j \in U$ such that for all $T \subseteq U$:

(a) $\sum_{j \in T} p_j \geq (v_i(U) - v_i(U \setminus T))$

(b) $\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T)$

Theorem

If a class of valuations admits balanced prices, then for any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$E \left[ \sum_{i=1}^{n} v_i(X_i) \right] \geq \frac{1}{2} \cdot E[OPT].$$
**Definition**

A valuation function $v_i$ admits $(\alpha, \beta)$-balanced prices if for every set of items $U \subseteq [m]$ there exist item prices $p_j$ for $j \in U$ such that for all $T \subseteq U$:

\[(a) \quad \sum_{j \in T} p_j \geq \frac{1}{\alpha} (v_i(U) - v_i(U \setminus T)) \]  
\[(b) \quad \sum_{j \in U \setminus T} p_j \leq \beta v_i(U \setminus T) \]

**Theorem**

If a class of valuations admits balanced prices, then for any distributions $D_1, \ldots, D_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

\[\mathbb{E}\left[ \sum_{i=1}^{n} v_i(X_i) \right] \geq \frac{1}{1 + \alpha \beta} \cdot \mathbb{E}[OPT].\]
Prices can be computed in polynomial time via LP relaxation (instead of integral optimum)
Prices can be computed in polynomial time via LP relaxation (instead of integral optimum)

Techniques also applicable for revenue maximization [Cai and Zhao, STOC 2017]
Combinatorial Auctions: Further Results

- Prices can be computed in polynomial time via LP relaxation (instead of integral optimum)

- Techniques also applicable for revenue maximization [Cai and Zhao, STOC 2017]

- Subadditive functions $v_i$ only admit $(\alpha, \beta)$-balanced prices for $\alpha \beta = \Omega(\log m)$ but better bounds via relaxation of balancedness [Dütting, K., Lucier, FOCS 2020]
Outline

1. Single Item

2. Balanced Prices for Combinatorial Auctions

3. Other Feasibility Structures
Matroid Prophet Inequality: Example

May assign...

≤ 100 seats in each of 10 blocks

≤ 500 seats in entire stadium
Matroid Prophet Inequality: Example

May assign...

\[ \leq 100 \text{ seats in each of 10 blocks} \]

\[ \leq 500 \text{ seats in entire stadium} \]

Laminar matroid
Matroid Prophet Inequality: Problem Statement

[Kleinberg and Weinberg, STOC 2012], somewhat implicit in earlier work

- $n$ buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- set of accepted buyers has to be independent in given matroid
- buyers arrive online, one after the other, reveal $v_i$, immediately decide whether to accept or to reject current buyer

- $v_i \sim \mathcal{D}_i$ independently
- buyers arrive in order $1, 2, \ldots, n$
Matroid Prophet Inequality: Problem Statement

[Kleinberg and Weinberg, STOC 2012], somewhat implicit in earlier work

- $n$ buyers of values $v_1, \ldots, v_n \in \mathbb{R}_{\geq 0}$
- set of accepted buyers has to be independent in given matroid
- buyers arrive online, one after the other, reveal $v_i$, immediately decide whether to accept or to reject current buyer

- $v_i \sim D_i$ independently
- buyers arrive in order $1, 2, \ldots, n$

Theorem (Kleinberg, Weinberg, STOC 2012)

*There is algorithm with*

$$\mathbb{E} [v(\text{ALG})] \geq \frac{1}{2} \mathbb{E} [v(\text{OPT})]$$
Generalized Balanced Prices

[Dütting, Feldman, K., Lucier, FOCS 2017], similar approach in [Kleinberg and Weinberg, STOC 2012]

Definition

A pricing rule \( p^v \) is \((\alpha, \beta)\)-balanced with respect to valuation profile \( v = (v_1, \ldots, v_n) \) if for all feasible \( x \) and all \( x' \) that are feasible "after" \( x \)

\[
\text{(a)} \quad \sum_i p^v_i(x_i) \geq \frac{1}{\alpha} (v(\text{OPT}(v)) - v(\text{OPT}(v | x)))
\]

\[
\text{(b)} \quad \sum_i p^v_i(x'_i) \leq \beta v(\text{OPT}(v | x))
\]

\( v(\text{OPT}(v | x)) \): Value that remains after allocating to \( x \)

\( v(\text{OPT}(v)) - v(\text{OPT}(v | x)) \): Value lost due to allocating to \( x \)
Generalized Balanced Prices
[Dütting, Feldman, K., Lucier, FOCS 2017], similar approach in [Kleinberg and Weinberg, STOC 2012]

Definition

A pricing rule \( p^v \) is \((\alpha, \beta)\)-balanced with respect to valuation profile \( v = (v_1, \ldots, v_n) \) if for all feasible \( x \) and all \( x' \) that are feasible “after” \( x \)

\[
\begin{align*}
\text{(a)} \quad & \sum_i p^v_i(x_i | x_{[i-1]}) \geq \frac{1}{\alpha} (v(OPT(v)) - v(OPT(v | x))) \\
\text{(b)} \quad & \sum_i p^v_i(x'_i | x_{[i-1]}) \leq \beta v(OPT(v | x))
\end{align*}
\]

\( v(OPT(v | x)) \): Value that remains after allocating to \( x \)

\( v(OPT(v)) - v(OPT(v | x)) \): Value lost due to allocating to \( x \)
Generalized Balanced Prices

[Dütting, Feldman, K., Lucier, FOCS 2017], similar approach in [Kleinberg and Weinberg, STOC 2012]

Definition

A pricing rule $p^v$ is $(\alpha, \beta)$-balanced with respect to valuation profile $v = (v_1, \ldots, v_n)$ if for all feasible $x$ and all $x'$ that are feasible “after” $x$

(a) $\sum_i p_i^v(x_i \mid x_{[i-1]}) \geq \frac{1}{\alpha} (v(OPT(v)) - v(OPT(v \mid x)))$

(b) $\sum_i p_i^v(x'_i \mid x_{[i-1]}) \leq \beta v(OPT(v \mid x))$

Theorem

If $p^v$ is $(\alpha, \beta)$-balanced for every $v$, then setting $p_i(x_i \mid y) = \frac{\alpha}{1+\alpha\beta} E_{\tilde{v} \sim D} [p_i^{\tilde{v}}(x_i \mid y)]$ achieves welfare at least $\frac{1}{1+\alpha\beta} E[v(OPT(v))]$. 
Example: Matroids
[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: $p_i(\text{accept} \mid A) = \text{OPT}(\tilde{v}, M/A) - \text{OPT}(\tilde{v}, M/(A \cup \{i\}))$, where

- $A$ is the set of buyers accepted so far
- $M/A$ is the matroid obtained by contracting $A$
Example: Matroids
[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: $p_i(\text{accept} \mid A) = \text{OPT}(\tilde{v}, \mathcal{M}/A) - \text{OPT}(\tilde{v}, \mathcal{M}/(A \cup \{i\}))$, where

- $A$ is the set of buyers accepted so far
- $\mathcal{M}/A$ is the matroid obtained by contracting $A$

One can show: $(1, 1)$-balanced
Example: Matroids

[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: \( p_i(\text{accept} \mid A) = \OPT(\tilde{v}, \mathcal{M}/A) - \OPT(\tilde{v}, \mathcal{M}/(A \cup \{i\})) \), where

- \( A \) is the set of buyers accepted so far
- \( \mathcal{M}/A \) is the matroid obtained by contracting \( A \)

One can show: \((1, 1)\)-balanced

Price for buyer \( i \): \( p_i(\text{accept} \mid A) = \frac{1}{2} \mathbf{E}_{\tilde{v}} \left[ \OPT(\tilde{v}, \mathcal{M}/A) - \OPT(\tilde{v}, \mathcal{M}/(A \cup \{i\})) \right] \).
Example: Matroids
[Kleinberg and Weinberg, STOC 2012], notation of [Dütting, Feldman, K., Lucier, FOCS 2017]

Pricing Rule: \( p_i(\text{accept} \mid A) = \OPT(\tilde{v}, \mathcal{M}/A) - \OPT(\tilde{v}, \mathcal{M}/(A \cup \{i\})) \), where

- \( A \) is the set of buyers accepted so far
- \( \mathcal{M}/A \) is the matroid obtained by contracting \( A \)

One can show: \((1, 1)\)-balanced

Price for buyer \( i \): \( p_i(\text{accept} \mid A) = \frac{1}{2} \mathbb{E}_{\tilde{v}} [\OPT(\tilde{v}, \mathcal{M}/A) - \OPT(\tilde{v}, \mathcal{M}/(A \cup \{i\}))]. \)

Prices have to depend on selections so far. [Feldman/Svensson/Zenklusen, SICOMP 2021]
Conclusion

■ General advantage:
  Showing that balancedness involves only full-information arguments

■ Can also be applied in prophet-secretary setting
  [Ehsani, Hajiaghayi, K., Singla, SODA 2018]

■ Sometime balanced prices give optimal guarantees for prophet inequalities
  (but not always)
General advantage:
  Showing that balancedness involves only full-information arguments

Can also be applied in prophet-secretary setting
  [Ehsani, Hajiaghayi, K., Singla, SODA 2018]

Sometime balanced prices give optimal guarantees for prophet inequalities (but not always)

Thank you!